Markov modeling of Stochastic Hybrid Systems

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***Abstract*—Hybrid systems are a useful abstraction for systems that have a combination of discrete and continuous dynamics. For typical examples of hybrid systems, there can be various sources of stochasticity. The source of stochasticity can be in the dynamics of the continuous states, the probabilistic switching between various modes of the system and the probabilistic resetting of the continuous state after switches. Such systems can be mathematically modeled by Discrete Time Stochastic Hybrid Systems (DTSHS). If the uncertainty in the initial condition of the stochastic hybrid system is specified by a probability distribution, it is useful to compute the probability distribution of the state of the system for some time in the future. This would allow one to quantify the probability of the system to be in an undesired or unsafe set. Such computations can be useful for probabilistic verification and validation of systems. In this paper, we discuss state space models for DTSHS and present computational methods to propagate probability distributions for DTSHS.**

I. Introduction

Hybrid systems consists of digital programs that interact with each other and with an analog environment. Examples include manufacturing controllers, aircraft management systems, unmanned aerial vehicles and robots. For typical examples of hybrid systems, there can be various sources of stochasticity. The dynamics of the continuous states can be inherently stochastic. For example, the wind acting on a helicopter or UAV can be modeled as a stochastic input. The resetting of the continuous states after switching between modes can be stochastic. For example, in the classic bouncing ball example, the velocity of the ball after the bounce can be reset in a stochastic manner - i.e., velocity of the ball after the bounce can be random due to the uneven surface of the ball. Switching between various modes of the hybrid system can also be probabilistic. This could be something that is inherent in the design of a digital controller - i.e., the controller switches between modes probabilistically.

Such hybrid systems with stochasticity can be mathematically modeled by Discrete Time Stochastic Hybrid Systems (DTSHS). This is an extension of the classic definition of hybrid systems - but with the stochastic nature of the continuous dynamics, switching and reset map accounted for.

An important problem relevant for hybrid systems is the problem of reachability - to find the set of reachable states of a hybrid system. This problem is decidable for linear hybrid automata (see [1], [2]). HyTech [2] is a model checker for linear hybrid automata that can be analyzed automatically by computing with polyhedral state sets. For nonlinear hybrid automata, it is possible to find conservative approximations for the set of reachable states using linear hybrid automata approximations (see [3]).

For stochastic hybrid systems, the equivalent problems of interest are probabilistic reachability analysis and probabilistic model checking. One could be interested to compute the probability of getting to a set of reachable states. Or one could be interested to decide whether the probability that a certain temporal logic formula holds true is below/above a certain threshold. This paper is one step in the direction of doing probabilistic reachability analysis/model checking for stochastic hybrid systems.

The approach used in this paper is use to finite state Markov chains to model the dynamics of the hybrid system. Such techniques have been made popular by the work of Dellnitz ([4]) and Froyland [5], where they use set-oriented methods to model continuous dynamical systems. The basic idea in these methods is to partition the continuous phase space into a finite number of sets. To construct the finite state Markov model, the index of each set is identified as the state of the Markov chain. Transition probabilities between different states of the Markov chain are interpreted as the probability of a typical point in one set to move to another set under one iteration of the map. We use the same approach here - except that we take into account the hybrid nature of the dynamics. i.e., we need to take into account the different dynamics for each mode of the hybrid system, the switching probabilities and the reset maps.

Discrete Time Stochastic Hybrid Systems have been investigated in the work by Abate et al. (see [6], [7], [8] and [9]). In these papers, the authors begin with a description of the system in terms of stochastic transition kernels and describe methods for approximate probabilistic reachability analysis. Whereas, in our paper, we define DTSHS using state space models and from these state space models, we define certain transfer operators that describe the evolution of the probability measure for the state of the hybrid system over time.

The rest of the paper is structured as follows. In Section II, we provide a definition of state space models for Discrete Time Stochastic Hybrid Systems. In Section III, we define certain transfer operators that describe the evolution of probability distributions on the hybrid state space. In Section IV, we discuss numerical methods for approximating the transfer operators. In V, we discuss two examples of DTSHS for which we use our computational methods.

II. Definition of Discrete Time Stochastic Hybrid Systems

We first define state space models for discrete time stochastic hybrid systems (DTSHS). We represent the discrete state space by $\mathcal{Q}$ and the continuous state is assumed to evolve in $\mathbb{R}^n$. Thus the state space of the hybrid system is given as

$$\mathcal{S} = \mathcal{Q} \times \mathbb{R}^n = \cup_i \{q_i\} \times \mathbb{R}^n. \quad (1)$$

The following definition formalizes the state space description of a discrete time stochastic hybrid system.

Definition 1. *The state space model for a discrete time stochastic hybrid system is a collection* $\mathcal{H} = (\mathcal{Q}, Init, T, L, R)$ *where*

- *(***modes***)* $\mathcal{Q} := \{q_1, q_2, ..., q_m\}$ *with* $m \in \mathbb{N}$*, represents the discrete state space;*
- *(***Initial uncertainty***)* $Init : \mathbb{B}(S) \rightarrow [0, 1]$ *is a probability measure on* S *for the initialization of the state variables.*
- *(***Flows***)* T *is a stochastic map that describes the dynamics of the continuous variables corresponding to each mode. The dynamics of the continuous variables corresponding to mode* q_i *is given as*

$$x(n+1) = T(q_i, x(n), \xi_i(n)), \quad (2)$$

  where $\xi_i(n)$ *is an i.i.d process with distribution* $\mathcal{N}_i$*.*
- *(***Switching function***)* S *is a switching probability function that gives the probability of switching between various modes.* $S(x, q_i, .)$ *is a probability measure on the discrete space space* $\mathcal{Q}$*. i.e.,* $S(x, q_i, q_j)$ *gives the probability of the system to jump from mode* q_i *to mode* q_j *given that the continuous state has value* x*.*
- *(***Reset Maps***)* R *is a stochastic map that probabilistically resets the values of the continuous state variables when a switch is made from mode* q_i *to mode* q_j*. The reset map is given as*

$$x(n+1) = R(q_i, q_j, x(n), \eta_{i,j}(n)), \quad (3)$$

  where $\eta_{i,j}(n)$ *is an i.i.d process with distribution* $\mathcal{W}_{i,j}$*.*

The execution of a state space model for the discrete time stochastic hybrid system over a finite time horizon $[0, N]$ is defined below.

Definition 2. *Consider the state space model for a DTSHS* $\mathcal{H} = (\mathcal{Q}, Init, T, L, R)$*. An execution of the model over a time horizon* $[0, N]$ *is given by the following algorithm:*

Set $k = 0$ *and extract a value* $(q(0), x(0))$ *according to the distribution* $Init$

while $k < N$ ***do***

Extract a value $q(k+1)$ *according to the probability distribution* $S(x(k), q(k), .)$*.*

Extract a value $\eta_{i,j}(k)$ *according to the distribution* $\mathcal{W}_{i,j}$*. Then compute*

$$x'(k+1) = R(q(k), q(k+1), x(k), \eta_{i,j}(k)), \quad (4)$$

$\{$*i is the index of the mode* $q(k)$ *and j is the index of the mode* $q(k+1)$*.*$\}$

Extract a value $\xi_j(k+1)$ *according to the distribution* $\mathcal{N}_j$*. Then compute*

$$x(k+1) = T(q(k+1), x'(k+1), \xi_j(k+1)), \quad (5)$$

$k \rightarrow k+1$

end while

Description: At a given time k, given the current mode $q(k)$ and continuous state $x(k)$, the mode at the next time-step $q(k+1)$ is probabilistically chosen according to the switching probability function $S(x(k), q(k), .)$. Once the mode at the next time-step is obtained, the continuous state is reset by the stochastic reset map. Note that there could be a reset even if $q(k+1) = q(k)$. This is the case for the bouncing ball example. In this example, there is only mode, but the state of the system can be reset if the position of the ball goes below zero. Once the continuous state is reset, it evolves one step according to the dynamics of the current mode $q(k+1)$, and this process is repeated N times.

III. Propagation of measures for Discrete Time Stochastic Hybrid Systems

An ordinary continuous dynamical system is formally a hybrid system with just one mode and no reset conditions. For such systems, the evolutions of densities on the state space are described in terms of so called transfer operators. The density could refer to a probably distribution that specifies the uncertainty in the current state of the dynamical system. Consider a dynamical system that evolves according to the deterministic map

$$x(k+1) = T(x(k)). \quad (6)$$

Let μ^k be a density that specifies the uncertainty in the state of the dynamical system at time k. Then the density μ^{k+1} has to satisfy the equality

$$\int_A \mu^{k+1}(x)dx = \int_{T^{-1}(A)} \mu^k(x)dx \text{ for all } A \subset \mathbb{R}^n. \quad (7)$$

The above equality is similar to a conservation of mass statement. i.e., the probability of the state $x(k+1)$ being in the set A is equal to the probability of finding the state $x(k)$ in the set $T^{-1}(A)$. The evolution of the density μ^k can be equivalently expressed using a linear transfer operator referred to as the Frobenius-Perron operator which is defined as the unique operator such that

$$\int_A [P]\mu(x)dx = \int_{T^{-1}(A)} \mu(x)dx \text{ for all } A \subset \mathbb{R}^n. \quad (8)$$

For a random dynamical system, evolving according to

$$x(k+1) = T(x(k), \xi(k)). \quad (9)$$

where ξ is a i.i.d. process, the Frobenius-Perron operator is defined as

$$\int_A [P]\mu(x)dx = \mathbb{E}_\xi \left[\int_{\mathbb{R}^n} \mu(x).\chi_A(T(x,\xi))dx \right], \quad \text{for all } A \subset \mathbb{R}^n. \tag{10}$$

For more details on the theory of these transfer operators, see [10]. Our objective is to define appropriate transfer operators for Discrete Time Stochastic Hybrid Systems. For this, we need to first define various operators corresponding to the flows, resets and switches of the hybrid system. With a slight abuse of notation, we use the same notation for probability measures and their corresponding probability distribution functions.

Definition 3. Flow Transfer Operators: *The Flow Transfer Operator corresponding to mode q_i, is defined as the Frobenius-Perron operator corresponding to the map $T(q_i,.,.)$. This is the unique operator $[P_i]$ such that*

$$\int_A [P_i]\mu(x)dx = \mathbb{E}_{\xi_i} \left[\int_{\mathbb{R}^n} \mu(x).\chi_A(T(q_i,x,\xi_i))dx \right], \quad \textit{for all } A \subset \mathbb{R}^n. \tag{11}$$

Note that eventhough the maps $T(q_i, x(k), \xi_i(k))$ are nonlinear, the Frobenius-Perron operators are linear operators, but infinite-dimensional.

Definition 4. Switching Transfer Operator: *The Switching Transfer Operator corresponding to the pair of modes q_i and q_j, is given as*

$$[L_{i,j}]\mu(x) = S(x, q_i, q_j).\mu(x). \tag{12}$$

Definition 5. Reset Transfer Operators: *The Reset Transfer Operator corresponding to the pair of modes q_i and q_j is given by the Frobenius-Perron operator corresponding to the reset map $R(q_i, q_j, ., .)$. This is given as*

$$\int_A [M_{i,j}]\mu(x)dx = \mathbb{E}_{\eta_{i,j}} \left[\int_{\mathbb{R}^n} \mu(x).\chi_A(R(q_i,q_j,x,\eta_{i,j}))dx \right], \quad \textit{for all } A \subset \mathbb{R}^n. \tag{13}$$

Let Γ^k be the probability distribution over the hybrid state space $\mathcal{S}$ for the current state of the hybrid system. We can define the following sub-probability measures on the continuous space R^n.

$$\mu_i^k(A) = \Gamma^k(q_i, A), \text{ for } i = 1, 2, ...m. \tag{14}$$

Note that $\mu_i^k(\mathbb{R}^n) \leq 1$. Hence it is a sub-probability measure. Also note that the probability of the state being in the set $A \subset \mathbb{R}^n$ (irrespective of the mode), is given by

$$\mu^k(A) = \sum_i \mu_i^k(A). \tag{15}$$

The evolution of the probability measure for the current state of the hybrid system can be described using a single transfer operator given as

$$\Gamma^{k+1} = \mathcal{P}\Gamma^k, \tag{16}$$

where

$$\Gamma^k = \begin{bmatrix} \mu_1^k \\ \mu_2^k \\ .... \\ .... \\ \mu_m^k \end{bmatrix}. \tag{17}$$

The Frobenius-Perron operator $\mathcal{P}$ for the hybrid system can be formally defined in terms of the Flow Transfer Operators, Switching Transfer Operators and Reset Transfer Operators as shown below.

Definition 6. Frobenius-Perron operator for DTSHS: *The Frobenius-Perron operator for the Discrete Time Stochastic Hybrid System $\mathcal{H}$ is the operator represented in block-operator form as*

$$\mathcal{P} = \begin{bmatrix} P_1M_{1,1}L_{1,1} & P_1M_{2,1}L_{2,1} & .... & P_1M_{m,1}L_{m,1} \\ P_2M_{1,2}L_{1,2} & P_2M_{2,2}L_{2,2} & .... & P_2M_{m,2}L_{m,2} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ P_mM_{1,m}L_{1,m} & P_mM_{2,m}L_{2,m} & .... & P_mM_{m,m}L_{m,m} \end{bmatrix}. \tag{18}$$

Each block of the above operator is a composition of the various flow, switching and reset transfer operators. Note that many of the blocks of the operator defined above can be zero because $L_{i,j}$ can be zero for many pairs i and j (i.e., there is no switching between modes q_i and q_j). Therefore it may be unnecessary to construct the whole operator defined above. It may also be infeasible to construct the whole operator because of the large number of modes or number of dimensions of the continuous state space. Thus it is useful to compute the action of the operator on measures algorithmically as described below.

The evolution of probability measures for the state of the DTSHS $\mathcal{H}$ over a finite time-horizon $[0, N]$ is given by the following algorithm:

Definition 7. Algorithm for Propagation of probability measures for a DTSHS:

Set $k = 0$ and set $\Gamma^0 = Init$.
while $k < N$ ***do***
  for $i = 1, 2, ..., m$ ***do***
    Get the sub-probability measures.

$$\mu_i^k(.) = \Gamma^k(q_i, .) \tag{19}$$

  end for
  for $i = 1, 2, ..., m$ ***do***
    for $j = 1, 2, ..., m$ ***do***

Compute the sub-probability measures

$$[\rho_{i,j}^k]^- = [L_{i,j}]\mu_i^k \tag{20}$$

Reset the sub-probability measures $[\rho_{i,j}^k]^-$

$$\rho_{i,j}^k = [M_{i,j}][\rho_{i,j}^k]^-. \tag{21}$$

end for
end for
for $i = 1, 2, ..., m$ ***do***
Compute the sub-probability measures $[\mu_i^{k+1}]^-$.

$$[\mu_i^{k+1}]^- = \sum_{j=1}^{m} \rho_{j,i}^k(.) \tag{22}$$

end for
for $i = 1, 2, ..., m$ ***do***
Compute (evolve sub-probability measures according to the flow of each mode)

$$\mu_i^{k+1} = [P_i]\,[\mu_i^{k+1}]^- \tag{23}$$

Set $\Gamma^{k+1}(q_i, .) = \mu_i^{k+1}(.)$.
end for
$k \to k+1$
end while

Description: The first step is to compute the fraction of the density μ_i^k that switches from mode q_i to mode q_j. This is represented by the sub-probability measures $[\rho_{i,j}^k]^-$ given in (20). The sub-probability measures $[\rho_{i,j}^k]^-$ need to be updated by the reset transfer operator. This is described in Equation (21). Once this is done, we compute the sub-probability measures $[\mu_i^{k+1}]^-$ as obtained by Equation (22). This step essentially adds all the parts of the densities μ_j^k that switched to mode q_i (after the reset). The next step is to propagate the sub-probability measures $[\mu_i^{k+1}]^-$ according to the dynamics of the mode q_i. This is done using the Flow Transfer Operator for each mode as shown in Equation (23).

IV. Numerical approximation of Transfer Operators for Discrete Time Stochastic Hybrid Systems

In [4], Dellnitz et al. describe set oriented numerical methods to construct finite dimensional approximations for the Frobenius-Perron operator corresponding to a continuous dynamical system. In this paper, we use the same techniques to construct approximations for the various transfer operators defined in the previous section. In this approach, the dynamics is modeled by a finite state Markov chain.

As described in [4], the transfer operator corresponding to a map T is constructed as follows. The state space is partitioned into a finite number of connected sets $\{A_1, A_2, ..., A_n\}$. To form the Markov model, each set A_i is identified with a state i of an n-state Markov chain. A $n \times n$ matrix P is constructed, where the entry P_{ij} is computed as

$$P_{ij} = \frac{m(A_i \cap T^{-1}A_j)}{m(A_i)}. \tag{24}$$

where m is the Lebesgue measure. The quantity P_{ij} can be interpreted as the probability that a typical point in A_i moves into A_j under one iteration of the map T. The quantity P_{ij} is computed by a Monte-Carlo approach. One randomly selects a large number of points $\{a_1, ..., a_N\} \subset A_i$ and sets $P_{ij} \approx \#\{a \in \{a_1, ..., a_N\} : T(a) \in A_j\}/N$. We construct such approximations for each one of the Flow Transfer Operators ($[P_i]$) and Reset Transfer Operators ($[M_{i,j}]$). There is no need to explicitly construct the Switching Transfer operator as its action is known exactly in terms of the switching probability function. The finite matrix approximations of the transfer operators are typically sparse and we use sparse matrix data structures to efficiently store them and compute matrix-vector products [11]. We have developed a software tool that take as input the specification of a DTSHS, constructs the various transfer operators and executes the algorithm for propagation of probability measures. A high-level flowchart of this software tool is shown in Figure 1. The most computationally intensive step in this software is the construction of the various transfer operators. The execution of measure propagating algorithm is comparatively efficient with the use of sparse matrix data structures.

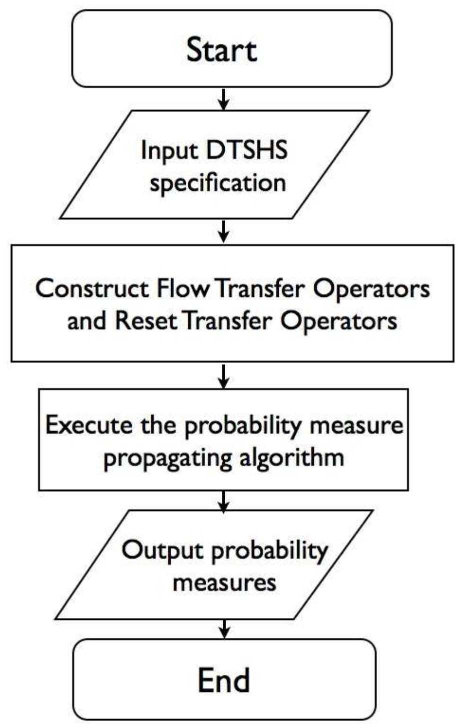

Fig. 1. Flowchart for software tool that propagates probability measures for Discrete Time Stochastic Hybrid Systems.

V. Examples

A. Thermostat

To simulate the temperature dynamics of thermostat regulated room, we use a hybrid system. In this example, the hybrid system has two modes - one when the 'heater' is off and one when the 'heater' is on. The dynamics of the temperature for these two modes are given as:

$$\textbf{OFF}: x(k+1) = x(k) - 0.25 + \xi(k) \tag{25}$$

$$\textbf{ON}: x(k+1) = x(k) - 0.25 + H + \xi(k). \tag{26}$$

The switching probability function for the system is defined as follows:

$$\begin{aligned}
S(x,0,1) &= \begin{cases} 0 \text{ if } x > x_{lmax} \\ \frac{x_{lmax}-x}{x_{lmax}-x_{lmin}} \text{ if } x_{lmin} \le x \le x_{lmax} \\ 1 \text{ if } x < x_{lmin} \end{cases} \\
S(x,0,0) &= 1 - S(x,0,1) \\
S(x,1,0) &= \begin{cases} 0 \text{ if } x < x_{hmin} \\ \frac{x-x_{hmin}}{x_{hmax}-x_{hmin}} \text{ if } x_{hmin} \le x \le x_{hmax} \\ 1 \text{ if } x > x_{hmax} \end{cases} \\
S(x,1,1) &= 1 - S(x,1,0).
\end{aligned} \tag{27}$$

The reset map for all mode transitions is identity. i.e., there is no change in the continuous state of the system after a mode transition. We set $H = 0.5$, $x_{lmin} = 48.0, x_{lmax} = 52.0, x_{hmin} = 58.0$ and $x_{hmax} = 62.0$. Figure 2 shows snapshots of the evolving probability measure at various times.

B. Bouncing Ball

In this example, the system has only one mode, but there is a reset condition when the position of the ball goes below zero. The reset map is made to be stochastic to capture effects of the uneven surface of the ball. The dynamics for the only mode of the system is given as:

$$\begin{aligned} x(k+1) &= x(k) + v(k).\Delta \\ v(k+1) &= v(k) - g\Delta \end{aligned} \tag{28}$$

where $x(k)$ is the position of the ball, $v(k)$ is the velocity and Δ is a sampling time-step. The reset map is identity when the position $x(k)$ is greater than zero. When $x(k)$ is less than zero, the position and velocity are reset as

$$\begin{aligned} x(k+1) &= -x(k) \\ v(k+1) &= -cv(k) + \eta(k). \end{aligned} \tag{29}$$

where $c = 0.85$. Figure 3 shows snapshots of the probability measure evolving according to the bouncing ball dynamics with the stochastic reset.

VI. CONCLUSION

We have presented a framework for propagating probability measures for the state of a discrete time stochastic hybrid system. In particular, we have defined certain transfer operators that describe the evolution of probability measures for stochastic hybrid systems. We have also described how to propagate these probability measures algorithmically. This is relevant for probabilistic reachability analysis/model checking. We start from the specification in terms of state space models for the stochastic hybrid system and then from this specification, we construct various transfer operators in a manner akin to that developed by Dellnitz et al. ([4]). A major computational challenge in this approach is the construction of the various transfer operators and we are currently investigating various computational techniques to efficiently approximate these transfer operators.

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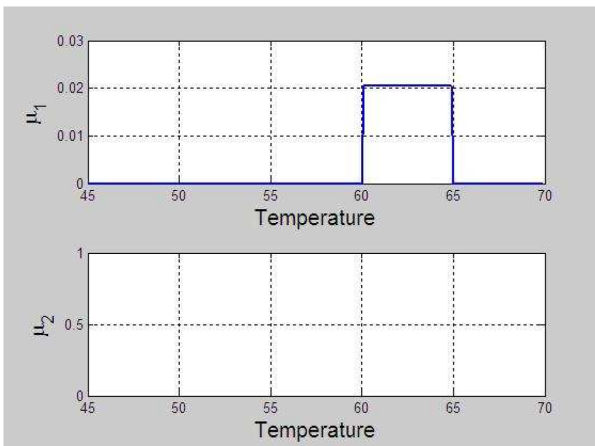
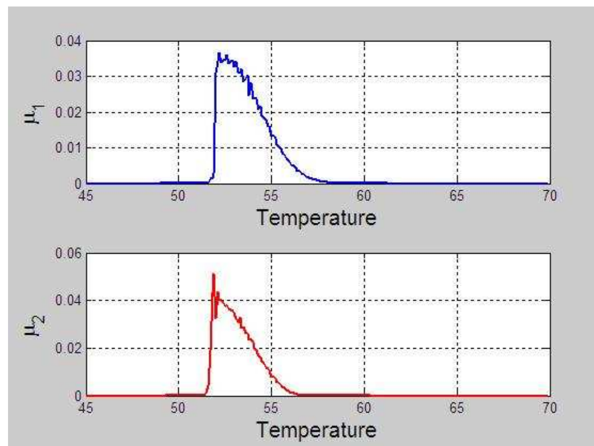
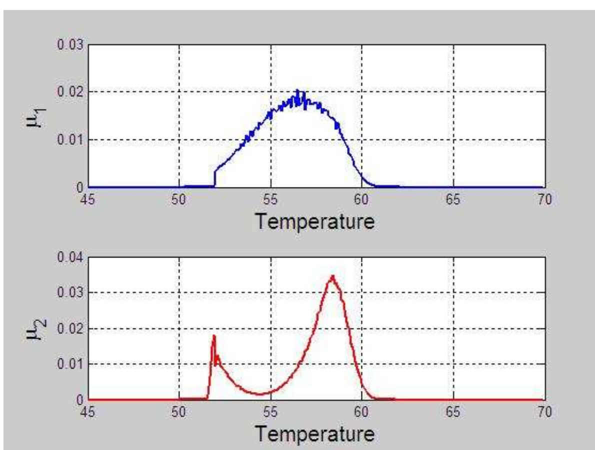


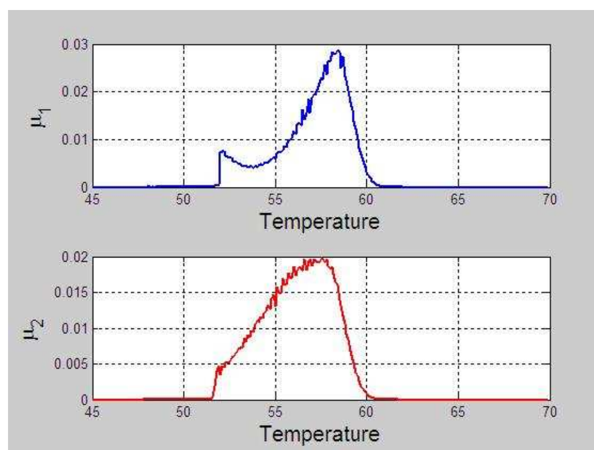
(a) Time,t=0



(b) Time,t=100

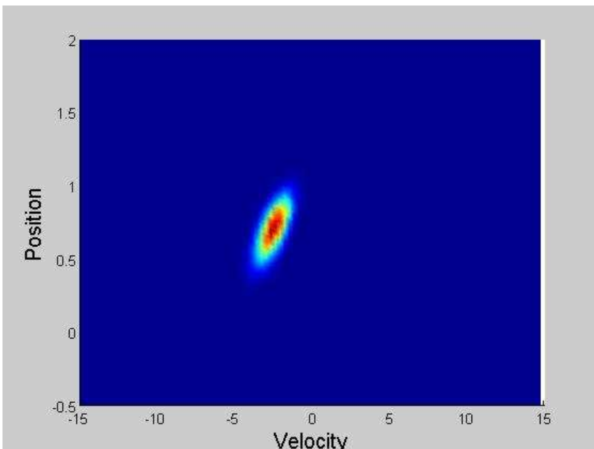


(c) Time,t=200

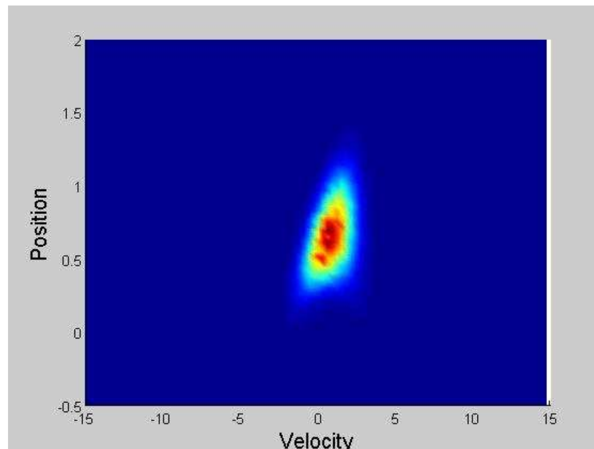


(d) Time,t=300

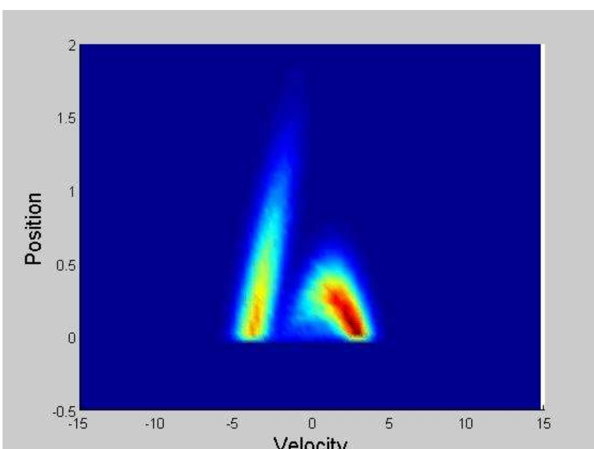
Fig. 2. Snapshots at various times of the probability measures for the thermostat example with probabilistic switching between the OFF and ON modes.



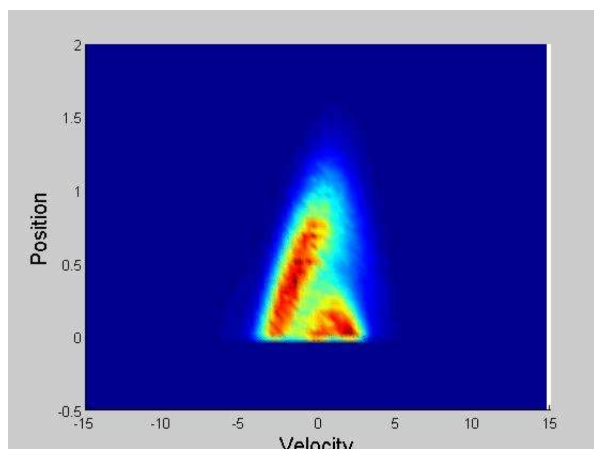
(a) Time,t=25



(b) Time,t=75



(c) Time,t=125



(d) Time,t=175

Fig. 3. Snapshots at various times of the probability measure evolving according to the bouncing ball dynamics with a stochastic reset.